Four Derivations of the Black Scholes PDE

by Fabrice Douglas Rouah
www.FRouah.com
www.Volopta.com

In this Note we derive the Black Scholes PDE for an option V, given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

We derive the Black-Scholes PDE in four ways.

1. By a hedging argument. This is the original derivation of Black and Scholes [1].
2. By a replicating portfolio. This is a generalization of the first approach.
3. By the Capital Asset Pricing Model. This is an alternate derivation proposed by Black and Scholes.
4. As a limiting case in continuous time of the Cox, Ross, Rubinstein [2] binomial model.

We also derive the PDE for the log-stock price instead of the stock price. To derive the PDE we assume the existence of three instruments

- A riskless bond B that evolves in accordance with the process $dB = rBdt$ where r is the risk-free rate.
- An underlying security which evolves in accordance with the Itō process $dS = \mu S dt + \sigma S dW$.
- A option V written on the underlying security which, by Itō's Lemma, evolves in accordance with the process

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\sigma S\frac{\partial V}{dS}\right)dW. \tag{2}$$

We have written $S = S(t), B = B(t), V = V(t)$ and $dW = dW(t)$ for notational convenience. We also assume the portfolios are self-financing, which implies that changes in portfolio value are due to changes in the value of the three instruments, and nothing else. Under this setup, any of the instruments can be replicated by forming a replicating portfolio of the other two instruments, using the correct weights.

1 Derivation of PDE by Hedging Argument

We set up a self-financing portfolio Π that is comprised of one option and an amount Δ of the underlying stock, such that the portfolio is riskless, i.e., that is insensitive to changes in the price of the security. Hence the value of the portfolio at time t is $\Pi(t) = V(t) + \Delta S(t)$. The self-financing assumption (see Section 2.1) implies that $d\Pi = dV + \Delta dS$ so we can write

$$\begin{aligned} d\Pi &= dV + \Delta dS \\ &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta \mu S\right) dt + \left(\sigma S \frac{\partial V}{\partial S} + \Delta \sigma S\right) dW. \end{aligned} \tag{3}$$

The portfolio must have two features. The first feature is that it must be riskless, which implies that the second term involving the Brownian motion dW is zero so that $\Delta = -\frac{\partial V}{dS}$. Substituting for Δ in Equation (3) implies that the portfolio follows the process

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

The second feature is that the portfolio must earn the risk free rate. This implies that the diffusion of the riskless portfolio is $d\Pi = r\Pi dt$. Hence we can write

$$\begin{aligned} d\Pi &= r\Pi dt \\ \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt &= r\left(V - \frac{\partial V}{\partial S} S\right) dt \end{aligned}$$

Dropping the dt term from both sides and re-arranging yields the PDE in Equation (1). The proportion of shares to be held, Δ, is delta, also called the hedge ratio. The derivation stipulates that in order to hedge the single option, we need to hold Δ shares of the stock. This is the principle behind delta hedging.

1.1 Original Derivation by Black and Scholes

In their paper, Black and Scholes [1] set up a portfolio that is slightly different: it is comprised of one share and $1/\Delta$ shares of the option. Hence, they define their portfolio to be $\Pi(t) = \theta V(t) + S(t)$. Similarly to Equation (3) they obtain

$$\begin{aligned} d\Pi &= \theta dV + dS \\ &= \left(\theta \frac{\partial V}{\partial t} + \theta \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\theta \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S\right) dt + \left(\theta \sigma S \frac{\partial V}{\partial S} + \sigma S\right) dW \end{aligned} \tag{4}$$

In order for the portfolio to be riskless, they set $\theta = -\left(\frac{\partial V}{\partial S}\right)^{-1}$. Substitute into Equation (4), equate with $d\Pi = r\Pi dt = r\left[\theta V + S\right] dt$ and drop the term involving μS to obtain

$$\left(\theta \frac{\partial V}{\partial t} + \frac{1}{2}\theta \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left[\theta V + S\right] dt.$$

Now drop dt from both sides and divide by θ to produce

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}.$$

Re-arranging terms produces the Black-Scholes PDE in Equation (1).

2 Derivation of PDE Using Pricing by Arbitrage

Pricing by arbitrage means in a complete market, all derivatives can be expressed in terms of a self-financing replicating strategy, and that this replicating strategy is unique. With this replicating strategy we can set up a replicating portfolio and use a risk neutral measure to calculate the value of the derivative.

2.1 Self Financing Trading Strategy

Given N assets with values $Z_1(t), \cdots, Z_N(t)$ at time t, a *trading strategy* is a N-dimensional stochastic process $a_1(t), \cdots, a_N(t)$ that represents the allocations into the assets at time t. The time-t value of the of the portfolio is $\Pi(t) = \sum_{i=1}^{N} a_i(t)Z_i(t)$. The trading strategy is *self-financing* if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds. This implies the strategy is self-financing if

$$d\Pi(t) = d\left(\sum_{i=1}^{N} a_i(t)Z_i(t)\right) = \sum_{i=1}^{N} a_i(t)dZ_i(t),$$

in other words, if

$$\Pi(t) = \Pi(0) + \sum_{i=1}^{N}\int_0^t a_i(u)dZ_i(u).$$

In the case of two assets the portfolio value is $\Pi(t) = a_1(t)Z_1(t) + a_2(t)Z_2(t)$ and the strategy (a_1, a_2) is self-financing if $d\Pi(t) = a_1(t)dZ_1(t) + a_2(t)dZ_2(t)$.

2.2 Arbitrage Opportunity

An *arbitrage opportunity* is a self-financing trading strategy that produces the following properties on the portfolio value:

$$\begin{aligned} \Pi(t) &\leq 0 \\ \Pr\left[\Pi(T) > 0\right] &= 1. \end{aligned}$$

This implies that the initial value of the portfolio (at time zero) is zero or negative, and the value of the portfolio at time T will be greater than zero with absolute certainty. This means that we start with a portfolio with zero value, or with debt (negative value). At some future time we have positive wealth, and since the strategy is self-financing, no funds are required to produce this wealth. This is a "free lunch."

2.3 Derivatives and Replication

The payoff $V(T)$ at time T of a derivative is a function of a risky asset. To rule out arbitrage we identify a self-financing trading strategy that produces the same payoff as the derivative, so that $\Pi(T) = V(T)$. The trading strategy is then a *replicating strategy* and the portfolio is a *replicating portfolio*. If a replicating strategy exists the derivative is *attainable*, and if all derivatives are attainable the economy is complete.

In the absence of arbitrage the trading strategy produces a unique value for the value $V(T)$ of the derivative, otherwise an arbitrage opportunity would exist. Not only that, at every time t the value of the derivative, $V(t)$ *must* be equal to the value of the replicating strategy, $\Pi(t)$, so that $\Pi(t) = V(t)$. Otherwise an arbitrage opportunity exists. Indeed, if $V(t) < \Pi(t)$ you could buy the derivative, sell the replicating strategy, and lock in an instant profit. At time T both assets would have equal value ($\Pi(T) = V(T)$) and the value of the bought derivative would cover the sold strategy. If $V(t) > \Pi(t)$ you could sell the derivative, buy the replicating strategy, and end up with the same outcome at time T. The technique of determining the value of a derivative by using a replicating portfolio is called *pricing by arbitrage*.

2.4 Derivation of the PDE by Replication

To replicate the derivative V we form a self-financing portfolio with the stock S and the bond B in the right proportion. Hence we need to use the replicating strategy $(a(t), b(t))$ to form the replicating portfolio $V(t) = a(t)S(t) + b(t)B(t)$ and determine the value of $(a(t), b(t))$. The self-financing assumption means that

$$dV = adS + bdB$$

where $a = a(t)$ and $b = b(t)$. Substituting for dV from Equation (2) and for dB and dS produces

$$\begin{aligned}\left(\frac{\partial V}{dt} + \mu S\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \left(\sigma S\frac{\partial V}{dS}\right)dW \\ = (a\mu S + brB)dt + a\sigma S dW\end{aligned} \tag{5}$$

Equating coefficients for dW implies that $a = \frac{\partial V}{\partial S}$. Substituting in Equation (5) produces

$$\begin{aligned}\frac{\partial V}{dt} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} &= brB \\ &= br\left(\frac{V - aS}{b}\right) \\ &= rV - rS\frac{\partial V}{\partial S}.\end{aligned}$$

Re-arranging terms produces the Black Scholes PDE in Equation (1).

2.4.1 Interpreting the Replicating Portfolio

The time-t Black-Scholes price of a call with time to maturity $\tau = T - t$ and strike K when the spot price is S is

$$\begin{aligned} V(S_t, K, T) &= S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \quad (6) \\ &= aS + bB \end{aligned}$$

where

$$d_1 = \frac{\log \frac{S}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$. It is easy to show, by differentiating the right-hand side of Equation (6), that $a = \frac{\partial V}{\partial S} = \Phi(d_1)$. Since $\Phi(d_1) > 0$ this implies that the replicating portfolio is long the stock, and since $\Phi(d_1) < 1$ the dollar amount of the long position is less than S, the spot price. Furthermore, since

$$bB = -Ke^{-r\tau}\Phi(d_2),$$

the replicating portfolio is short the bond. Finally, since $e^{-r\tau}\Phi(d_2) < 1$, the dollar amount of the short position is less than K, the strike price.

2.5 Replicating the Security

In the original Black-Scholes derivation of Section (1) we are in fact replicating the bond B with the option V and the security S. In the arbitrage derivation of Section (2.4) we are replicating the option with the security and the bond. We can also replicate the security with the bond and the option, and obtain the Black-Scholes PDE. We form the portfolio $S = B + \phi V$ where ϕ needs to be determined. Applying the self-financing assumption implies that

$$dS = dB + \phi dV$$

so we can write

$$\begin{aligned} \mu S dt + \sigma S dW &= rBdt + \phi\left(\frac{\partial V}{dt} + \mu S\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \quad (7) \\ &\quad + \phi\left(\sigma S\frac{\partial V}{dS}\right)dW. \end{aligned}$$

This implies that $\phi = \left(\frac{\partial V}{dS}\right)^{-1}$. We can write Equation (7) as

$$\mu S dt = rBdt + \left(\frac{\partial V}{dS}\right)^{-1}\left(\frac{\partial V}{dt} + \mu S\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

Drop the dt terms from both sides, substitute $B = S - V\left(\frac{\partial V}{dS}\right)^{-1}$ to obtain

$$\mu S = r\left(S - \frac{V}{\frac{\partial V}{dS}}\right) + \left(\frac{\partial V}{dS}\right)^{-1}\left(\frac{\partial V}{dt} + \mu S\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)$$

so that

$$\mu S\frac{\partial V}{dS} = rS\frac{\partial V}{dS} - rV + \left(\frac{\partial V}{dt} + \mu S\frac{\partial V}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right).$$

Cancelling terms and rearranging yields the PDE in Equation (1).

3 Derivation Using the CAPM

This derivation is included in the original derivation of the PDE by Black and Scholes [1].

3.1 The CAPM

The Capital Asset Pricing Model (CAPM) stipulates that the expected return of a security i in excess of the risk-free rate is

$$E\left[r_i\right] - r = \beta_i \left(E\left[r_M\right] - r\right)$$

where r_i is the return on the asset, r is the risk-free rate, r_M is the return on the market, and

$$\beta_i = \frac{Cov\left[r_i, r_M\right]}{Var\left[r_M\right]}$$

is the security's beta.

3.2 The CAPM for the Assets

In the time increment dt the expected stock price return, $E\left[r_S dt\right]$ is $E\left[\frac{dS_t}{S_t}\right]$, where S_t follows the diffusion $dS_t = rS_t dt + \sigma S_t dW_t$. The expected return is therefore

$$E\left[\frac{dS_t}{S_t}\right] = rdt + \beta_S \left(E\left[r_M\right] - r\right)dt. \tag{8}$$

Similarly, the expected return on the derivative, $E\left[r_V dt\right]$ is $E\left[\frac{dV_t}{V_t}\right]$, where V_t follows the diffusion in (2), is

$$E\left[\frac{dV_t}{V_t}\right] = rdt + \beta_V \left(E\left[r_M\right] - r\right)dt. \tag{9}$$

3.3 The Black-Scholes PDE from the CAPM

The derivative follows the diffusion

$$dV_t = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial t}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(dS\right)^2$$

Divide by V_t on both sides to obtain

$$\frac{dV_t}{V_t} = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\frac{dS_t}{S_t}\frac{S_t}{V_t},$$

which is

$$r_V dt = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\frac{S_t}{V_t} r_S dt. \tag{10}$$

Drop dt from both sides and take the covariance of r_V and r_M, noting that only the second term on the right-hand side of Equation (10) is stochastic

$$Cov\left[r_V, r_M\right] = \frac{\partial V}{\partial S}\frac{S_t}{V_t} Cov\left[r_S, r_M\right].$$

This implies the following relationship between the beta of the derivative, β_V, and the beta of the stock, β_S

$$\beta_V = \left(\frac{\partial V}{\partial S}\frac{S_t}{V_t}\right)\beta_S.$$

This is Equation (15) of Black and Scholes [1]. Multiply Equation (9) by V_t to obtain

$$\begin{aligned} E\left[dV_t\right] &= rV_t dt + V_t \beta_V \left(E\left[r_M\right] - r\right) dt \qquad (11) \\ &= rV_t dt + \frac{\partial V}{\partial S} S_t \beta_S \left(E\left[r_M\right] - r\right) dt. \end{aligned}$$

This is Equation (18) of Black and Scholes [1]. Take expectations of the second line of Equation (2), and substitute for $E\left[dS_t\right]$ from Equation (8)

$$E\left[dV_t\right] = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}\left[rS_t dt + S_t \beta_S \left(E\left[r_M - r\right]\right) dt\right] + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt. \tag{12}$$

Equate Equations (11) and (12), and drop dt from both sides. The term involving β_S cancels and we are left with the PDE in Equation (1)

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0.$$

4 Derivation as a Limit of the Binomial Model

The stock price at time t is S_t. Define $u = e^{\sigma\sqrt{dt}}$ and $d = e^{-\sigma\sqrt{dt}}$. At time $t + dt$ the stock price moves up to $S_{t+dt}^u = uS_t$ with probability

$$p = \frac{e^{rdt} - d}{u - d},$$

or down to $S_{t+dt}^d = dS_t$ with probability $1 - p$. Risk-neutral valuation of the derivative gives rise to the following relationship (see Cox, Ross, and Rubinstein [2])

$$Ve^{rdt} = pV_u + (1 - p) V_d = p\left(V_u - V_d\right) + V_d \tag{13}$$

where $V = V(S_t), V_u = V\left(S^u_{t+dt}\right)$ and $V_d = V\left(S^d_{t+dt}\right)$. Now take Taylor series expansions of V_u, V_d, e^{rdt}, u, and d up to order dt. We have

$$\begin{aligned} V_u &\approx V + \frac{\partial V}{\partial S}\left(S^u_{t+dt} - S_t\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(S^u_{t+dt} - S_t\right)^2 + \frac{\partial V}{\partial t}dt \\ &= V + \frac{\partial V}{\partial S}S_t(u-1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S_t^2(u-1)^2 + \frac{\partial V}{\partial t}dt. \end{aligned} \tag{14}$$

Similarly

$$V_d \approx V + \frac{\partial V}{\partial S}S_t(d-1) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S_t^2(d-1)^2 + \frac{\partial V}{\partial t}dt. \tag{15}$$

The other expansions are

$$\begin{aligned} e^{rdt} &\approx 1 + rdt, \\ u &\approx 1 + \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt, \\ d &\approx 1 - \sigma\sqrt{dt} + \frac{1}{2}\sigma^2 dt. \end{aligned}$$

Note that $(u-1)^2 = (d-1)^2 = \sigma^2 dt$. This implies that we can write

$$\begin{aligned} p(V_u - V_d) &= p(u-d)\frac{\partial V}{\partial S}S_t \\ &= \left(rdt + \sigma\sqrt{dt} - \frac{1}{2}\sigma^2 dt\right)\frac{\partial V}{\partial S}S_t. \end{aligned} \tag{16}$$

Substitute (15) and (16) in Equation (13) and cancel terms to produce

$$V(1 + rdt) = rS_t\frac{\partial V}{\partial S}dt + V + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial t}dt.$$

Cancel V from both sides and divide by dt to obtain the Black Scholes PDE in Equation (1).

5 The PDE in Terms of the Log Stock Price

Recall the Black Scholes PDE in (1)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Consider the transformation $x = \ln S$. Apply the chain rule to the first-order derivative to obtain

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial V}{\partial x}\frac{1}{S}.$$

The second order derivative is a little more complicated and also requires the product rule

$$\begin{aligned}
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial V}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial V}{\partial x}\right) \\
&= -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S}\frac{\partial^2 V}{\partial x^2}\frac{\partial x}{\partial S} = -\frac{1}{S^2}\frac{\partial V}{\partial x} + \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2}.
\end{aligned}$$

Substitute into the PDE (1) to obtain the PDE in terms of the log stock price

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial x} - rV = 0.$$

The transformation creates a PDE with constant coefficients rather than coefficients that depend on S.

References

[1] Black, F., and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, Vol 81, No. 3, pp. 637-654.

[2] Cox, J.C., Ross, S.A., and M. Rubinstein (1979). "Option Pricing: A Simplified Approach." *Journal of Financial Economics*, Vol. 7, pp. 229-263.